

Mikhail Zaidenberg

HYPERBOLIC SURFACES in \mathbb{P}^3 : EXAMPLES

(after a joint work with B. Shiffman)

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1. Generalities

DEFINITION: On any complex space X there exists a unique *KOBAYASHI PSEUDOMETRIC* $k_X : X \times X \rightarrow \mathbb{R}$ such that:

- (i) If $X = \Delta$ is the unit disc then k_X is the Poincaré metric;
- (ii) any holomorphic map $\varphi : \Delta \rightarrow X$ is a contraction: $\varphi^*(k_X) \leq k_\Delta$, and
- (iii) k_X is the largest pseudometric on X which satisfies (i) and (ii).

In fact, k_X is the integrated form of the *ROYDEN INFINITESIMAL PSEUDOMETRIC*:

$$\forall x \in X, \forall v \in T_x X, \\ K_X(x, v) = \inf \{ r^{-1} \mid \exists \varphi \in \text{HOL}(\Delta_r, X) : 0 \mapsto x, \partial/\partial x \mapsto v \}$$

REMARK: Actually, any holomorphic map $\varphi : Y \rightarrow X$ of complex spaces is a contraction: $\varphi^*(k_X) \leq k_Y$.

DEFINITION: A complex space X is called *KOBAYASHI HYPERBOLIC* if k_X is a metric, i.e. $k_X(p, q) = 0 \Leftrightarrow p = q$.

EXAMPLES: 1. The Kobayashi pseudometric on a projective space \mathbb{P}^n (on an affine space \mathbb{C}^n , on a complex torus \mathbb{C}^n/Λ , respectively) vanishes identically.

2. Any bounded domain in \mathbb{C}^n is hyperbolic. In particular, the Teichmüller space $X = T_{g,n}$ is hyperbolic, and the Kobayashi metric k_X coincides with the Teichmüller metric (**Royden 1971**).

Geometric function theory
on hyperbolic complex spaces

Schottky-Landau THEOREM: $\mathbb{C} \setminus \{0, 1\}$ is hyperbolic.

Thus 'X is hyperbolic' means that the Schottky-Landau Theorem holds for X.

Brody-Kiernan-Kobayashi-Kwack THEOREM: Let X be a compact complex space. Then the following are equivalent:

(i) X is Kobayashi hyperbolic;

(ii) **(Picard Theorem)**

X does not contain entire curves i.e. $\forall f \in \text{HOL}(\mathbb{C}, X), f = \text{const}$;

(iii) **(Big Picard Theorem)**

$$\forall f \in \text{HOL}(\Delta \setminus \{0\}, X) \exists \bar{f} \in \text{HOL}(\Delta, X), \bar{f}|(\Delta \setminus \{0\}) = f;$$

(iv) **(Montel Theorem)** the space $\text{HOL}(\Delta, X)$ is compact.

REMARK: Actually, X is hyperbolic \implies for any complex space Y, the space $\text{HOL}(Y, X)$ is compact.

STABILITY THEOREM: (a) **(Brody [Br])** If a compact complex subspace $X \subset Z$ of a complex space Z is hyperbolic then there exists a hyperbolic neighborhood $U \supset X$ in Z. Consequently, any complex subspace X' in Z sufficiently close to X is hyperbolic.

(b) **(Zaidenberg [Za2])** Suppose Z is compact and X, X' are divisors in Z such that X, $Z \setminus X$ are hyperbolic. If X' is sufficiently close to X then as well X', $Z \setminus X'$ are hyperbolic.

THEOREM: If $f : Y \rightarrow X$ is a covering then f is a local isometry: $f^*(K_X) = K_Y$.

COROLLARY: X is hyperbolic \Leftrightarrow Y is hyperbolic.

REMARK: In particular, the Kobayashi metric on a Riemann surface X of non-exceptional type coincides with the Poincaré metric of X.

COROLLARY ('The Covering Trick'): Let $H \subset \mathbb{P}^{n+1}$ be a hypersurface and $\pi : H \rightarrow \mathbb{P}^n$ a projection ramified over $X \subset \mathbb{P}^n$. If H is hyperbolic then $\mathbb{P}^n \setminus X$ is hyperbolic as well.

2. Kobayashi Problems

PROBLEM (Kobayashi 1970 [Ko1]): *Let X be a (very) general hypersurface in \mathbb{P}^n of degree d . Is it true that for any $d \gg 1$, X resp., $\mathbb{P}^n \setminus X$ is hyperbolic? Is this true already for $d \geq 2n - 1$ resp., $d \geq 2n + 1$?*

Hyperbolicity of complements

The latter bound in the Kobayashi Problem is due to the famous

Borel LEMMA (E. Borel 1898; Bloch 1926): $\mathbb{P}^n \setminus \{2n + 1 \text{ hyperplanes in general position}\}$ *is hyperbolic.*

In fact, this bound is optimal:

PROPOSITION (Zaidenberg [Za1]): *The complement of a general hypersurface of degree $d \leq 2n$ in \mathbb{P}^n is not hyperbolic.*

THEOREM (Siu-Yeung [SY2], $d \geq 10^6$; El Goul [EG]): *For a very generic curve $C \subset \mathbb{P}^2$ of degree $d \geq 13$ the complement $\mathbb{P}^2 \setminus C$ is hyperbolic.*

Thus the hyperbolicity of the complement of a generic plane curve of degree d is still to be established for $5 \leq d \leq 12$. Starting with dimension 3, the Kobayashi problem on hyperbolicity of complements is open. As for examples, an irreducible plane curve of degree 30 with hyperbolic complement was constructed by **Azukawa-Suzuki 1980**. Later on, examples in smaller degrees were found by **Grauert-Peternell, Carlson-Green, Adachi-Suzuki, Dethloff-Schumacher-Wong, Dethloff-Zaidenberg, Berteloot-Duval**, e.a. Examples of projective hypersurfaces in any dimension with hyperbolic complements were suggested by **Masuda-Noguchi 1996, Fujimoto 2000** e.a.

THEOREM: (a) **(Zaidenberg [Za2])** *For every $d \geq 5$ there exists an irreducible smooth curve C in \mathbb{P}^2 of degree d with hyperbolic complement. However, there is no such curve of degree $d \leq 4$.*

(b) **(Green [Gr1]; see also [Ba, Za2, ErSo])** *Let $D = \bigcup_{i=1}^{2n+1} D_i$ be a union of hyperbolic hypersurfaces in general position in \mathbb{P}^n . Then for any (smooth) hypersurface $D' \subset \mathbb{P}^n$ close enough to D , the complement $\mathbb{P}^n \setminus D'$ is hyperbolic.*

(c) **(Shiffman-Zaidenberg [ShZa1])** *There exist algebraic families of hyperbolic hypersurfaces of degree $d = (2n - 1)^2 + 2n$ in \mathbb{P}^n with hyperbolic complements.*

In particular (c) provides algebraic families of curves of degree 13 in \mathbb{P}^2 , of surfaces of degree 31 in \mathbb{P}^3 , and so forth, whose complements are complete hyperbolic and hyperbolically embedded in projective space. These examples are of Fermat-Waring type (see below).

Hyperbolicity of projective hypersurfaces

THEOREM (McQuillan [MQ], $d \geq 36$; Demailly-El Goul [DEG], $d \geq 21$):
A very general surface of degree $d \geq 21$ in \mathbb{P}^3 is Kobayashi hyperbolic.

Thus the hyperbolicity of a generic surface in \mathbb{P}^3 of degree d is still to be established for $5 \leq d \leq 20$. Starting with dimension $n = 4$ the Kobayashi problem on hyperbolicity of a generic hypersurface in \mathbb{P}^n is open¹.

REMARK: By the Stability Theorem, if a degree d hypersurface X in \mathbb{P}^n is hyperbolic then so is any hypersurface X' in \mathbb{P}^n of the same degree, sufficiently close to X . Thus such hypersurfaces are parametrized by an open, in the classical topology, subset \mathcal{H} in \mathbb{P}^{N-1} , where $N = \binom{n+d}{d}$. The Kobayashi Problem asks, in particular, whether \mathcal{H} contains a Zariski open subset. What can be said about the boundary of \mathcal{H} ?

Methods employed:

1. Value distribution theory; the Nevanlinna-Cartan theorem
 (Green 1975, Masuda-Noguchi 1996, McQuillan-Brunella 1998, e.a.)
2. Meromorphic connections
 (Siu 1987, Nadel 1989, Demailly-El Goul 1997, e.a.)
3. Jet differentials/jet metrics
 (Bloch 1926, Green-Griffith 1979, Grauert 1989, Siu-Yeung 1996, Demailly-El Goul 1997, Dethloff-Lu 1996, e.a.)
4. Algebraic (multi)foliations
 (McQuillan 1997, Demailly-El Goul 1997, e.a.)

¹**Y.-T. Siu** (unpublished) announced a positive solution.

Examples
of hyperbolic surfaces in \mathbb{P}^3 of degree d :

Below $a, a_i \in \mathbb{C}$ are generic.

Brody-Green 1977, $d = 2k \geq 50$:

$$X_0^d + X_1^d + X_2^d + X_3^d + a_1(X_0X_1)^k + a_2(X_2X_3)^k = 0.$$

Masuda-Noguchi 1996, $d = 3e \geq 24$:

$$X_0^d + X_1^d + X_2^d + X_3^d + a(X_0X_1X_2)^e = 0,$$

$d = 4e \geq 28$:

$$X_0^d + X_1^d + X_2^d + X_3^d + a(X_0X_1X_2X_3)^e = 0,$$

and similar examples of hyperbolic hypersurfaces in \mathbb{P}^n for all $n \geq 4$; they are of degree $d \geq 192$ for $n = 4$ and $d \geq 784$ for $n = 5$.

Khoai 1996, $d \geq 22$:

$$X_0^d + X_1^d + X_2^d + X_3^d + aX_0^{\alpha_0}X_1^{\alpha_1}X_2^{\alpha_2} = 0$$

where $\alpha_i \geq 7$, $\alpha_0 + \alpha_1 + \alpha_2 = d$.

Nadel 1989, $d = 6e + 3 \geq 21$:

$$X_0^{6e}(X_0^3 + aX_1^3) + X_1^d + X_2^d + X_3^{6e}(X_3^3 + aX_1^3) = 0.$$

El Goul 1996, $d \geq 14$:

$$X_0^{d-2}(X_0^2 + aX_1^2) + X_1^d + X_2^d + X_3^{d-2}(X_3^2 + aX_1^2) = 0.$$

Siu-Yeung 1996, **Demailly-El Goul 1997**, $d \geq 11$:

$$X_0^d + X_1^d + X_2^d + X_3^{d-2}(a_0X_0^2 + a_1X_1^2 + a_2X_2^2 + a_3X_3^2) = 0.$$

J. Duval 2000 [Du1], **Fujimoto 2000**² [Fu], $d = 2k \geq 8$:

$$Q(X_0, X_1, X_2)^2 - P(X_2, X_3) = 0,$$

where P, Q are general homogeneous forms of degree d resp., k . The idea behind this example is quite simple. Let S be a surface in \mathbb{P}^3 as in the example. A resolution $f : S' \rightarrow \mathbb{P}^1$ of the meromorphic map $(X_2 : X_3) : S \dashrightarrow \mathbb{P}^1$ has reducible general fibers and so admits a Stein factorization $S' \xrightarrow{\varphi} \Gamma \xrightarrow{2:1} \mathbb{P}^1$, where Γ is ramified over the zeros of P . As $k \geq 4$ then Γ and every fiber of φ are irreducible curves of genera ≥ 2 .

²Also **Shirosaki 2000**, $d = 2k \geq 10$ [Shr1, Shr2].

Hence by Picard's Theorem any holomorphic map $\mathbb{C} \rightarrow S'$ is constant, and as well any holomorphic map $\mathbb{C} \rightarrow S$ is. Thus by Brody's Theorem X is hyperbolic.

Examples of hyperbolic hypersurfaces in any dimension were given by **Masuda-Noguchi 1996, Khoai 1996, Siu-Yeung 1997, Fujimoto 2000 and Shiffman-Zaidenberg 2000**. The best degree asymptotic in these examples is achieved by the following Fermat-Waring type hypersurfaces.

THEOREM (Siu-Yeung [SY1], $d = 16(n-1)^2$, Shiffman-Zaidenberg [ShZa2]): *Let $d \geq (m-1)^2$, $m \geq 2n-1$. Then for generic linear functions h_1, \dots, h_m on \mathbb{C}^{n+1} , the hypersurface $X \subseteq \mathbb{P}^n$ with equation*

$$\sum_{j=1}^m h_j^d = 0$$

is hyperbolic. In particular there exist algebraic families of hyperbolic hypersurfaces in \mathbb{P}^n of degree $d = 4(n-1)^2$.

3. Symmetric powers of curves as hyperbolic hypersurfaces in \mathbb{P}^3 and \mathbb{P}^4

(after [ShZa1], [CiZa])

DEFINITION: A smooth projective curve C is called *HYPERELLIPTIC* resp., *BIELLIPTIC* if there exists a $2 : 1$ morphism $C \rightarrow \mathbb{P}^1$ resp., $C \rightarrow E$ where E is an elliptic curve.

LEMMA: (a) *The n -ths symmetric power $C^{(n)}$ of a generic genus g curve C is hyperbolic if and only if $g \geq 2n-1$.*

(b) *The symmetric square $C^{(2)}$ of a curve C is hyperbolic iff C is neither hyperelliptic nor bielliptic. In particular, $C^{(2)}$ is hyperbolic for a genus $g \geq 3$ curve C with general moduli.*

REMARKS: (a) A genus 2 curve is hyperelliptic.

(b) A non-hyperelliptic genus 3 curve C is a smooth plane quartic, and vice versa; it is bielliptic iff the group $\text{AUT}(C) \subset \text{PSL}(3; \mathbb{C})$ is of even order, that is, iff this group contains an involution.

THEOREM [ShZa1]: *Let C be a genus $g \geq 3$ curve with general moduli, $C^{(2)}$ be its symmetric square embedded in \mathbb{P}^5 , and S be a general projection of $C^{(2)}$ in \mathbb{P}^3 . Then S is a hyperbolic surface of degree $d \geq 16$.*

Example of degree 16:

Let C be a non-bielliptic smooth plane quartic; e.g.

$$C = \{x^4 - xz^3 - y^3z = 0\} \subset \mathbb{P}^2.$$

Then the symmetric square $C^{(2)}$ of C is a hyperbolic surface in the symmetric square $S^{(2)}\mathbb{P}^2$ of \mathbb{P}^2 . In turn, $S^{(2)}\mathbb{P}^2$ can be embedded in \mathbb{P}^5 as the cubic 4-fold

$$rst + 4uvw - r^2w - s^2u - t^2v = 0$$

under $\varphi : S^{(2)}\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$,

$$\{(x : y : z), (x' : y' : z')\} \mapsto (xy' + x'y : yz' + y'z : xz' + x'z : xx' : yy' : zz').$$

This provides an embedding $\varphi : C^{(2)} \hookrightarrow \mathbb{P}^5$ of degree 16. A generic projection to \mathbb{P}^3 of the image $\varphi(C^{(2)})$ with center at a generic line in \mathbb{P}^5 is a hyperbolic surface in \mathbb{P}^3 of degree 16.

This surface is singular. By the Stability Theorem its general small deformation is a smooth hyperbolic surface of degree 16 in \mathbb{P}^3 .

REMARK: There are examples (**Kaliman-Zaidenberg** [KaZa]) of non-hyperbolic singular projective surfaces X with a smooth hyperbolic normalization $\tilde{X} \rightarrow X$.

Hyperbolic 3-folds in \mathbb{P}^4 birational to the symmetric cube of a curve

THEOREM (Ciliberto-Zaidenberg [CiZa]): *Let C be a curve of genus $g \geq 7$ with general moduli, and let $C^{(3)}$ be its symmetric cube embedded in \mathbb{P}^n , $n \geq 7$. If T is a generic projection of $C^{(3)}$ in \mathbb{P}^4 then T is a hyperbolic 3-fold in \mathbb{P}^4 . The same conclusion holds for certain special embeddings $C^{(3)} \hookrightarrow \mathbb{P}^7$, where C is a general plane quintic ($g = 6$). The minimal degree of such a hyperbolic 3-fold in \mathbb{P}^4 is 125.*

4. Deformation method and more hyperbolic surfaces in \mathbb{P}^3

(after [ShZa3, ShZa4])

In Example 3 below we give a simple construction of hyperbolic surfaces in \mathbb{P}^3 of any given degree $d \geq 8$. We need the following notion.

DEFINITION: Let M be a complex manifold. A *BRODY CURVE* in M is an entire curve $\varphi : \mathbb{C} \rightarrow M$ such that $\|\varphi'(\zeta)\|$ is bounded by $\|\varphi'(0)\| = 1$, where the norm is computed with respect to a hermitian metric on M .

THEOREM (Brody [Br]) *A compact complex manifold M is hyperbolic iff it does not contain Brody curves.*

DEFORMATION METHOD: Let $\{X_t\} = \langle X_0, X_\infty \rangle$ be a linear pencil of degree d surfaces in \mathbb{P}^3 generated by X_0, X_∞ , where X_∞ is general and X_0 is singular with a double curve $\Gamma \subseteq \text{sing } X_0$ i.e., X_0 has two branches at general points of Γ . Then sufficiently small deformations X_t of X_0 are hyperbolic provided that X_0 , while not hyperbolic, satisfies the following 'hyperbolic non-percolation' property with respect to a certain divisor $D \supseteq \Gamma \cap X_\infty$ on Γ , which will be precised later on.

DEFINITION: Let D be a divisor on Γ . We say that $X_0 \setminus \Gamma$ has the property of *HYPERBOLIC NON-PERCOLATION through D* if there is no Brody curve $\varphi : \mathbb{C} \rightarrow (X_0 \setminus \Gamma) \cup D$.

INDICATION FOR THE DEFORMATION METHOD:

Assume that, for a sequence $t_n \rightarrow 0$, the surfaces X_{t_n} are not hyperbolic. By Brody's Reparametrization Lemma, there exists a sequence of Brody curves $f_n : \mathbb{C} \rightarrow X_{t_n}$ converging to a Brody curve $f : \mathbb{C} \rightarrow X_0$. By Hurwitz's Theorem then either

- $f(\mathbb{C}) \subset \Gamma \setminus \{p_j\}$, where the p_j are the points of Γ where X has 3 or more local branches, or
- $f(\mathbb{C}) \subset (X_0 \setminus \Gamma) \cup D$ with D consisting of $\Gamma \cap X_\infty$ and the points of Γ where X_0 is unbranched.

Indeed, suppose that $f(\mathbb{C})$ passes through a point $p \in \Gamma \setminus X_\infty$, and X_0 has k branches at p . Locally in a small neighborhood U of p , X_t can be given by an equation of the form $\varphi_1 \cdot \dots \cdot \varphi_k + t\varphi_\infty = 0$, where $\varphi_i(p) = 0$ and φ_∞ does not vanish in U . Let Δ_ε be a small disc in \mathbb{C} with center a such that $f_n(\Delta_\varepsilon) \subseteq U$ for all $n \gg 1$ and $f(a) = p$. Then $\varphi_i \circ f_n$ does not vanish in Δ_ε while $\varphi_i \circ f$ does. By Hurwitz's Theorem $\varphi_i \circ f \equiv 0$ ($i = 1, \dots, k$), as needed.

Consequently, if

- $\Gamma \setminus \{p_j\}$ is hyperbolic, and
- $X_0 \setminus \Gamma$ has the property of hyperbolic non-percolation through D ,

then all sufficiently small deformations X_t of X_0 are hyperbolic.

EXAMPLE 1 [ShZa3]: We let X_0 be the union of 15 hyperplanes $\{l_j = 0\}$, $j = 1, \dots, 15$, in general position in \mathbb{P}^3 and $X_\infty = 3Q$, where $Q = \{q = 0\}$ is a generic quintic surface. We consider the linear pencil $\langle X_0, X_\infty \rangle$ of surfaces X_t in \mathbb{P}^3 , where

$$X_t = \left\{ \prod_{j=1}^{15} l_j + tq^3 = 0 \right\}.$$

Then the surface X_t is hyperbolic for all sufficiently small $t \neq 0$.

CONJECTURE: *A general small deformation of 6 planes in general position in \mathbb{P}^3 is a hyperbolic sextic surface.*

Actually, it is enough to show that the complement of the union L of general 5 lines in \mathbb{P}^2 has hyperbolic non-percolation property with respect to $D = L \cap C$, where C is a general sextic curve.

EXAMPLE 2 [ShZa3]: There exists an octic surface X_0 in \mathbb{P}^3 with a double curve Γ such that the normalization of X_0 is smooth and is a simple abelian surface A . Then for a general octic surface X_∞ in \mathbb{P}^3 , all sufficiently small deformations $X_t \in \langle X_0, X_\infty \rangle$ ($t \neq 0$) of X_0 are hyperbolic.

EXAMPLE 3 [ShZa4]: We let X_0 be the union of two general cones $CF_i = \langle a_i, F_i \rangle$ in \mathbb{P}^3 with vertices a_i ($a_1 \neq a_2$) over generic plane curves F_i of degree $d_i \geq 4$, $i = 1, 2$. Then for a general surface X_∞ in \mathbb{P}^3 of degree $d = d_1 + d_2$, all sufficiently small deformations $X_t \in \langle X_0, X_\infty \rangle$ ($t \neq 0$) of X_0 are hyperbolic. In suitable coordinates $(Z_0 : \dots : Z_3)$ in \mathbb{P}^3 , X_t is given by the equation

$$f_1(Z_0, Z_1, Z_2)f_2(Z_1, Z_2, Z_3) + tf_\infty(Z_0, Z_1, Z_2, Z_3) = 0,$$

where $F_i = \{f_i = 0\}$, $i = 1, 2$, and $X_\infty = \{f_\infty = 0\}$.

INDICATION: We let $\Gamma = CF_1 \cap CF_2$ be the double curve of X_0 . By our genericity assumption, the projection $\pi_{a_i} : \Gamma \rightarrow F_i$ with center a_i has degree $d_j \geq 4$ ($j \neq i$) and only simple ramifications. Hence every fiber of $\pi_{a_i}|_\Gamma$ contains at least 3 points. If nearby surfaces X_t were not hyperbolic, then one would find a sequence of Brody curves $f_{t_n} : \mathbb{C} \rightarrow X_{t_n}$ converging to a Brody curve $f : \mathbb{C} \rightarrow X_0$. Let $f(\mathbb{C}) \subseteq CF_i$ ($i \in \{1, 2\}$). Since F_i has genus $g_i \geq 3$, by Picard's Theorem $\pi_{a_i} \circ f : \mathbb{C} \rightarrow F_i$ is constant. Thus $f(\mathbb{C}) \subseteq l$, where $l \cong \mathbb{P}^1$ is a projective line through a_i and a point $b \in F_i$.

Moreover, by Hurwitz's Theorem, $f(\mathbb{C})$ does not meet Γ except, maybe, at points of $X_\infty \cap \Gamma$. But a general surface X_∞ does not pass through the ramification points of the projection $\pi_{a_i} : \Gamma \rightarrow F_i$ and meets any fiber of this projection in at most one point. It follows that $\Gamma \setminus X_\infty$ contains at least 3 points of l . Hence by Picard's Theorem $f : \mathbb{C} \rightarrow l \setminus (\Gamma \setminus X_\infty)$ must be constant. This is a contradiction.

EXAMPLE 4 (Duval 2004 [Du2]) : The same deformation-nonpercolation methods, applied iteratively in a clever way, provide an example of a degree 6 hyperbolic surface in \mathbb{P}^3 .

5. Algebraic hyperbolicity

DEFINITION: A projective variety is said to be *ALGEBRAICALLY HYPERBOLIC* if it does not contain rational curves or images of abelian varieties (in particular, it does not contain elliptic curves).

REMARK: Clearly, if X is hyperbolic then it is algebraically hyperbolic. As for the converse, no (projective) counter-example is known. Moreover, basing on the **Bloch Conjecture** (see below), **Green** and **Griffiths** proposed the following one.

Green-Griffiths CONJECTURE 1979 :

If X is a projective surface of general type then the image of any entire curve $f : \mathbb{C} \rightarrow X$ is contained in a closed algebraic curve on X .

Thus, $f(\mathbb{C})$ has to be contained in a rational or elliptic curve on X . Clearly, the number of these curves of any given degree is finite. **Bogomolov's CONJECTURE 1975** says that the total number of such curves in X is as well finite. Some partial results were obtained by **Bogomolov 1977**, **Lu-Yau 1990**, **Lu-Miyaoka 1997**, **McQuillan 1998**, **Brunella 1999 e.a.** For instance (**Grant 1986**), the Green-Griffiths conjecture is true for surfaces with irregularity $q(X) \geq 2$.

It is well known and elementary that on any hypersurface of degree $d \leq 2n - 3$ in \mathbb{P}^n there are projective lines. The number of these lines on a general hypersurface of degree $2n - 3$ is finite (e.g., there are 27 lines on a smooth cubic surface in \mathbb{P}^3). A smooth quartic surface S in \mathbb{P}^3 (and more generally, any algebraic $K3$ surface) if does not contain lines, however contains rational curves and a family of elliptic curves (**Mumford-Bogomolov-Mori-Mukai 1981**). The rational (respectively, elliptic) curves on S arise e.g., as sections by tritangent (respectively, bitangent) planes of S .

THEOREM (Clemens [Cl], Ein [Ei], Xu [Xu], Voisin [Vo], Pacienza [Pa]):

Let X be a very generic hypersurface of degree d in \mathbb{P}^n .

(a) *If $d \geq 2n - 1$ then X is algebraically hyperbolic. In particular, a very generic surface in \mathbb{P}^3 of degree $d \geq 5$ is.*

(b) *For $n \geq 4$ and $d \geq 2n - 2$, X does not contain rational curves, and for $n \geq 6$ and $d = 2n - 3$, the only rational curves on X are lines.*

REMARKS: (a) By the **Clemens CONJECTURE**, a general quintic threefold $X \subset \mathbb{P}^4$ ($d = 2n - 3$) contains only a finite number of rational curves of any given degree. The number of these curves is predicted by the Mirrow Symmetry.

(b) If, for a given (n, d) , there exists a hyperbolic degree d hypersurface in \mathbb{P}^n then, clearly, a very generic such hypersurface is algebraically hyperbolic. Thus examples of hyperbolic hypersurfaces provide an alternative potential approach to (a) in the above theorem. However, e.g., for $n = 3$, $d = 5$, so far we are lacking such examples.

6. Bloch Conjecture

Bloch 1926 formulated a conjecture (and an idea of the proof) concerning hyperbolicity of subvarieties of abelian varieties $A = \mathbb{C}^n/\Lambda$ and of the complements of divisors $A \setminus D$. The simplest result in this direction is the following one.

THEOREM (Green 1978 [Gr2]) : *A closed subvariety X of a compact complex torus \mathbb{C}^n/Λ is hyperbolic iff it does not contain shifted subtori.*

Indeed, according to Brody's Theorem, \mathbb{C}^n/Λ is not hyperbolic iff there exists a Brody curve $f : \mathbb{C} \rightarrow X$. Clearly, the covering Brody curve $\tilde{f} : \mathbb{C} \rightarrow \mathbb{C}^n$ (w.r.t. the Euclidean metric on \mathbb{C}^n) is an affine linear map, hence the closure of the image $f(\mathbb{C})$ in X contains a shifted subtorus.

REMARK: A generic complex torus (resp., a generic Abelian variety, resp., a generic Jacobian variety) is simple.

COROLLARY: *Let $T = \mathbb{C}^n/\Lambda$ be a simple complex torus, i.e. it does not contain any proper subtorus of a positive dimension. Then any proper subvariety V of T is hyperbolic. For instance, the theta-divisor Θ in a simple abelian variety T is hyperbolic.*

Many subsequent efforts were done to fix the Bloch Conjecture in the full generality (Lang 1966, Ax 1972, Ochiai 1977, Noguchi 1977, 1996, Noguchi-Winkelmann 2003, Green 1978, Green-Griffith 1979, Kawamata 1980, R. Kobayashi 1991, D. Abramovich 1994, McQuillan 1995, Siu-Yeung 1996, 2003, Demailly 1996, Dethloff-Lu 1996, e.a.)

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A P P E N D I X

For reader's convenience we place below, as an appendix, the survey article [Za3] which was published in an edition with a limited access.

Hyperbolicity in Projective Spaces

Mikhail Zaidenberg

Université Grenoble I

Institut Fourier de Mathématiques

38402 St Martin d'Hères-cedex, France

*To Professor Shoshichi Kobayashi
on the occasion of his sixtieth birthday*

In 1970 Sh. Kobayashi posed the following problems [Ko1]:

Let D be a generic hypersurface of degree d in \mathbb{P}^n , where d is large enough with respect to n .

I *Is it true that D is hyperbolic?*

II *Is it true that the complement $\mathbb{P}^n \setminus D$ is hyperbolic and, moreover, hyperbolically embedded into \mathbb{P}^n ? Is this true for $d \geq 2n + 1$?*

For $n = 2$ (starting with $d = 4$) the answer to **I** is classically known to be positive, while for $n \geq 3$ the problem is open.

The answer to **II** is unknown even for $n = 2$. It is positive for $n = 1, d \geq 3$, and this is equivalent to the Montel Theorem.

Here we present a survey on the Kobayashi's Problems. Of course, it does not pretend to be either exhaustive or original.

I The compact case

Let $\mathbb{P}_{n,d} = \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$, be the projective space whose points parametrize (not necessarily reduced) hypersurfaces of degree d in \mathbb{P}^n . Let $\mathcal{H}_{n,d} \subset \mathbb{P}_{n,d}$ be the subset corresponding to hyperbolic hypersurfaces. To precise the meaning of "genericity" in **I** one could ask *whether $\mathcal{H}_{n,d}$ contains a Zariski open subset of $\mathbb{P}_{n,d}$ for $d \gg n$?* Or, more generally, *whether the complement $\mathbb{P}_{n,d} \setminus \mathcal{H}_{n,d}$ is contained in a countable union of hypersurfaces in $\mathbb{P}_{n,d}$ for $d \gg n$?*

It is known that $\mathcal{H}_{n,d}$ is open (but probably empty) in the classical Hausdorff topology of $\mathbb{P}_{n,d}$ for any $n, d \in \mathbb{N}$. This follows from the Brody's Stability Theorem [Br], or, to be more precise, from the following version of it [Za1,4]:

Theorem I.1 *Let M be a complex manifold and X a compact analytic subset of M . If X is hyperbolic, then there exists a neighborhood U of X in M , which is hyperbolically embedded into M . Therefore, any compact analytic subset X' in M close enough to X is hyperbolic as well.*

In particular, if $f : M \rightarrow S$ is a proper holomorphic surjection onto a complex space S , then the subset of points in S that correspond to the hyperbolic fibers of f is open.

Let us give a sketch of proof.

Let h be a fixed Hermitian metric on M . An entire curve $f : \mathbb{C} \rightarrow M$ is called a *Brody curve* iff f is a contraction with respect to the Euclidean metric in \mathbb{C} and the metric h on M (i.e. $|df(z)|_h \leq 1 \forall z \in \mathbb{C}$), and $|df(0)|_h = 1$.

Let Δ_r be the open disc in \mathbb{C} of radius r centered at the origin endowed with the metric rh_r , where h_r is the Poincaré metric in Δ_r . It is easily seen that the Euclidean metric in \mathbb{C} is the limit of the metrics rh_r as $r \rightarrow \infty$. A holomorphic curve $f : \Delta_r \rightarrow M$ is called a *Brody curve* iff f is a contraction with respect to the metrics rh_r in Δ_r and h in M , and $|df(0)|_h = 1$. By the Arzelà-Ascoli Theorem any sequence $f_n : \Delta_n \rightarrow M$

of Brody curves, whose images are contained in the same relatively compact subset of M , has a subsequence converging to a Brody curve $f : \mathbb{C} \rightarrow M$.

Let $\{U_n\}$ be a fundamental sequence of (relatively compact) neighborhoods of the hyperbolic compact analytic subset $X \subset M$. Suppose that there is no $n \in \mathbb{N}$ such that U_n is hyperbolically embedded into M . That means that the inequality $K_{U_n} \geq ch$ for the Kobayashi-Royden pseudometric K_{U_n} on U_n does not hold for any constant $c > 0$; in particular, it does not hold for $c = \frac{1}{n}$. By the definition of the Kobayashi-Royden pseudometric there exists a sequence $h_n : \Delta_n \rightarrow U_n$ of holomorphic curves such that $|dh_n(0)| > 1$. By the Brody Reparametrization Lemma [Br] there exists a sequence of Brody curves $f_n : \Delta_n \rightarrow U_n$, where $f_n(z) = h_n \circ \alpha_n(r_n z)$ for some $r_n < 1$ and $\alpha_n \in \text{Aut}(\Delta_n)$. Passing to a convergent subsequence, one obtains a limit Brody curve $f : \mathbb{C} \rightarrow \bigcap U_n = X$, that contradicts our assumption on hyperbolicity of X . \circ

So, the hyperbolicity of a hypersurface in \mathbb{P}^n is stable under small deformations of the coefficients of the defining equation. More generally, in any component of the Hilbert scheme of projective varieties of a given degree and dimension, the set of points which correspond to hyperbolic varieties is open in the usual topology. We do not know *when this set is non-empty; whether, being non-empty, it must contain a Zariski open subset, or at least an algebraic subvariety of small enough codimension.*

For $n = 3$ R. Brody and M. Green [BrGre] gave examples of one-parametric families of hyperbolic surfaces in \mathbb{P}^3 of any even degree $d = 2k \geq 50$. Namely, the surfaces

$$D_{d,t} = \{x_0^{2k} + x_1^{2k} + x_2^{2k} + x_3^{2k} + t(x_0x_1)^k + t(x_0x_2)^k = 0\}$$

(deformations of the Fermat surfaces $F_{3,d} = D_{d,0}$) are hyperbolic for all but a finite number of values of $t \in \mathbb{C}$. This means that for $d = 2k \geq 50$ the set $\mathcal{H}_{3,d}$ is non-empty and contains a quasi-projective rational curve $C = \{D_{d,t}\}$ (together with some small classical neighborhood of it, as follows from the Stability Theorem).

It is unknown whether for any $n \geq 4$ there exists a hyperbolic hypersurface in \mathbb{P}^n . J. Noguchi (private communication) supposed that the Brody-Green construction should be available as well in higher dimensions, and at least for $n = 4$.

Notice that the Newton polyhedron of the Fermat hypersurface $F_{n,d}$ of degree d in \mathbb{P}^n is the standard simplex in \mathbb{R}^{n+1} ; the monomials in the Fermat equation correspond to its vertexes. Additional monomials in the Brody-Green example correspond to the middle points of some edges of this simplex. So the defining polynomials are fewnomials i.e., they contain few monomials with respect to their degrees.

Definition. Let us say that a hypersurface $D = \{p(x_0, \dots, x_n) = 0\}$ of degree d in \mathbb{P}^n is *k-almost simplicial* if any monomial of p corresponds to a lattice point in \mathbb{R}^{n+1} with one of coordinates $\geq d - k$ (that is this point is situated in a k -neighborhood of some vertex of the n -simplex $\{x_0 + \dots + x_n = d\}$ in \mathbb{R}_+^{n+1}).

The following statement is due to A. Nadel [Na]; its proof is based on the Y.-T. Siu's version of the value distribution theory for holomorphic curves in a complex manifold in presence of a meromorphic connection.

Theorem I.2 *For arbitrary $e \geq 3$ in the projective space of all k -almost simplicial surfaces in \mathbb{P}^3 of degree $d = 6e + 3 > 4k + 10$ there exists a quasiprojective subvariety of dimension $4\binom{k+4}{4} - 1$, which consists of hyperbolic smooth surfaces. In particular, $\mathcal{H}_{3,d}$ is non-empty for any $d = 6e + 3 \geq 21$.*

Definition. Let us say that a complex Hermitian manifold (X, h) is *Brody hyperbolic* iff it does not contain any Brody curve $\mathbb{C} \rightarrow X$, and *Picard hyperbolic* iff it does not contain any non-constant entire curve $\mathbb{C} \rightarrow X$.

The Picard Theorem says that $\mathbb{P}^1 \setminus \{3 \text{ points}\}$ is Picard hyperbolic. The Brody Theorem [Br] states that for a compact manifold X all three notions of hyperbolicity (i.e., the Kobayashi hyperbolicity, the Brody hyperbolicity and the Picard hyperbolicity) are equivalent.

M. Green [Gre4] noticed that a Brody curve $\mathbb{C} \rightarrow \mathbb{T}^n$ in a complex torus $\mathbb{T}^n = \mathbb{C}^n / \Lambda$, where Λ is a maximal rank lattice in $\mathbb{C}^n = \mathbb{R}^{2n}$, lifts to an isometric affine embedding $\mathbb{C} \rightarrow \mathbb{C}^n$. Therefore, a closed subvariety $X \subset \mathbb{T}^n$ is (Brody) hyperbolic iff it does not contain any shifted subtorus. The same is true for any compact complex parallelizable manifold [HuWi].

More generally, Sh. Kobayashi [Ko2] established the following fact.

Theorem I.3 *Let (X, h) be a Hermitian manifold with non-positive holomorphic sectional curvature, and $f : \mathbb{C} \rightarrow X$ be a Brody curve. Then f is an isometric immersion, and its image is totally geodesic.*

Problem I.1 *Let the conditions of the above theorem be fulfilled. Is it true that the closure $\overline{f(\mathbb{C})}$ in X contains the image of a complex torus by a non-constant holomorphic map, or at least any compact complex submanifold of positive dimension?*

We notice that the rational curve \mathbb{P}^1 and the simple complex tori are the only known examples of compact complex manifolds with totally degenerate Kobayashi pseudodistances that are minimal in this class, i.e. that contain no closed subvarieties with this property to be *completely non-hyperbolic*. This motivates the following

Definition. A compact complex space is said to be *algebraically hyperbolic* if it contains no image of a complex torus by a non-constant holomorphic map.

In particular, an algebraically hyperbolic variety contains no rational or elliptic curve. Clearly, a hyperbolic complex space is also algebraically hyperbolic.

Problem I.2 *Does algebraic hyperbolicity imply (Brody) hyperbolicity, at least for projective varieties? In other words, is it true that a compact complex space (a complex projective variety) which possesses a Brody curve, must contain the image of a complex torus under a non-constant holomorphic map?*

The following recent result of J.-P. Demailly and B. Shiffman [DemSh] could be considered as an approximation to the positive answer.

Theorem I.3 *Let X be a smooth projective variety, S a Stein manifold with $\dim S \leq \dim X$, $f : S \rightarrow X$ a holomorphic map, T a finite subset of S and m a natural number. Then there exists an exhaustive sequence $\Omega_1 \subset \dots \subset \Omega_k \subset \dots$ of Runge domains in S and a sequence of holomorphic maps $f_k : \Omega_k \rightarrow X_k$ such that, for any $k \in \mathbb{N}$, $\dim X_k = \dim S$ and at each point $s \in T$ the m -jet of f_k coincides with the m -jet of f . If S is an affine algebraic manifold, then f_k can be chosen to be regular.*

As a corollary, one gets the following algebraic definition of the Kobayashi-Royden pseudometric K_X on a projective variety X :

$$K_X(v) = \inf \{ K_{\tilde{C}}(v) \mid v \in TC \},$$

where C runs over the set of all algebraic curves in X such that $v \in TX$ is a tangent vector to C , and $K_{\tilde{C}}$ is the Poincaré metric of the normalization \tilde{C} of C . Furthermore,

the Kobayashi pseudodistance $k_X(x, y)$ on X coincides with its algebraic analogue $d_X(x, y)$ suggested by J. Noguchi. Roughly speaking, the chains of holomorphic discs in the definition of the Kobayashi pseudodistance are replaced by chains of algebraic curves, and the hyperbolic metrics of these curves are used instead of the Poincaré metric in the disc.

An approach to Kobayashi's Problem I is to divide it into two parts: the above Problem I.2 on the equivalence of the Brody hyperbolicity and the algebraic hyperbolicity for projective varieties, as the first part, and as the second one the following

Problem I.3 *Is it true that a generic projective hypersurface of a large enough degree in \mathbb{P}^n is algebraically hyperbolic?*

For $n = 3$ the positive answer follows from the next recent result of Geng Xu [Xu], which was conjectured by J. Harris and yields a precision of an earlier one due to H. Clemens.

Theorem I.4 *For any algebraic curve on a generic surface $D \in \mathbb{P}_{3,d}$ of degree $d \geq 5$ in \mathbb{P}^3 the following estimate holds:*

$$g(\tilde{C}) \geq \frac{d(d-3)}{2} - 2 \geq 3,$$

where $g(\tilde{C})$ is the genus of the normalization \tilde{C} of C . This bound is sharp, and for $d \geq 6$ the curves of the minimal genus are sections of D by tritangent planes.

Therefore, for $d \geq 5$ a generic surface of degree d in \mathbb{P}^3 does not contain any rational or elliptic curve, and so is algebraically hyperbolic.

Observe that on a smooth quartic surface in \mathbb{P}^3 , and moreover on any K3-surface, there exist a rational curve and a linear pencil of elliptic curves (see [GreGri] and [MoMu]). Thus such a surface is not algebraically hyperbolic. This shows that the bound $d \geq 5$ above is sharp.

The proof of Theorem I.4 involves the Brill-Noether Theorem, and thus the meaning of "genericity" in its formulation is more extended than the genericity in Zariski sense. Namely, let $\mathcal{AH}_{n,d} \subset \mathbb{P}_{n,d}$ be the set of all algebraically hyperbolic hypersurfaces. Then by Theorem I.4 for $d \geq 5$ the complement $\mathbb{P}_{3,d} \setminus \mathcal{AH}_{3,d}$ consists of a countable number of proper algebraic subvarieties of $\mathbb{P}_{3,d}$. There is no information about their mutual position. In particular, the following problem seems to be important.

Problem I.3 *Is the locus $\mathbb{P}_{3,d} \setminus \mathcal{AH}_{3,d}$ closed in $\mathbb{P}_{3,d}$ in the usual topology?*

Supposing this locus is not closed, there should exist a sequence of non-algebraically hyperbolic surfaces D_k in \mathbb{P}^3 converging to an algebraically hyperbolic surface D_0 . By the stability of hyperbolicity, D_0 is not Brody hyperbolic; indeed, otherwise for k large enough D_k would be hyperbolic as well, and therefore algebraically hyperbolic. So, if the answer to Problem I.3 were negative, then also the answer to Problem I.2 would be negative, and D_0 would be an example of an algebraically hyperbolic surface which is not hyperbolic and hence contains a Brody entire curve $\mathbb{C} \rightarrow D_0$.

A generic (in Zariski sense) hypersurface of degree $d \leq 2n - 3$ in \mathbb{P}^n contains a projective line (in particular, a smooth cubic surface in \mathbb{P}^3 contains exactly 27 lines), thus is not algebraically hyperbolic.

Question. *What is the maximal number $d = d(n)$ such that $\mathbb{P}_{n,d} \setminus \mathcal{AH}_{n,d}$ contains a Zariski open subset of $\mathbb{P}_{n,d}$?*

By the above remarks we have that $d(3) = 4$ and $d(n) \geq 2n - 3$.

It is worthwhile also mentioning the following well known problems:

Whether hyperbolicity (resp. algebraic hyperbolicity), or even measure hyperbolicity of a compact complex manifold implies that it is a projective variety of general type?

The positive answer is known in the case of surfaces (see [GreGri], [MoMu]).

A weaker property that could serve as a bridge between hyperbolicity and algebraic hyperbolicity, is *algebraic degeneracy*.

Definition. One says that a complex space X has the property of *algebraic degeneracy* iff the image of any non-constant entire curve $\mathbb{C} \rightarrow X$ lies in a proper closed complex subspace of X . We mention *strong algebraic degeneracy*, if this subspace is the same for all such curves.

Perhaps, it is worthwhile also to specify this notion by restricting the class of curves under consideration to Brody curves.

The Bloch Conjecture, proven by T. Ochiai, Y. Kawamata, and also by M. Green and P. Griffiths, R. Kobayashi (see [RKo] for references), states that *an irregular projective variety X (i.e. a variety with the irregularity $q(X) = h^{1,0}(X) > \dim X$) has the property of algebraic degeneracy*. The above restriction was weakened in the case of surfaces of general type to $q(X) \geq 2$ by C. Grant [Gra1] (see also [Gra2], [HuWi], [Lu] and St. Lu's report in this volume for some related results).

Another property, close to algebraic hyperbolicity, is finiteness of the number of non-hyperbolic (resp. non-algebraically hyperbolic) proper subvarieties. In the surface case this is finiteness of the number of rational and elliptic curves, that was proved by F. Bogomolov [Bo] for projective surfaces of general type with $c_1^2 > c_2$ (see also [Lu]). H. Clemens conjectured that the number of rational curves of any given degree d on a generic quintic threefold in \mathbb{P}^4 is finite, that was verified by N. Katz for $d \leq 7$ (see [Xu]).

II The non-compact case

Denote by $\mathcal{HE}_{n,d}$ the subset of $\mathbb{P}_{n,d}$ consisting of all hypersurfaces of degree d in \mathbb{P}^n with hyperbolically embedded complements. Then $\mathcal{HE}_{n,d}$ is non-empty for any $d \geq 2n + 1$; indeed, it contains the union $C_{n,d}$ of d hyperplanes in general position. This fact (modulo Kiernan's criterion of hyperbolic embedding [Ki2]) goes back to E. Borel, A. Bloch, A. Cartan and J. Dufresnoy (see [KiKo] for references). It was reproved many times, for instance by M. Green [Gre2], E. Babets [Ba] and others.

The bound $d \geq 2n + 1$ for $\mathcal{HE}_{n,d}$ being non-empty should be sharp. It is sharp for $n = 2$; indeed, M. Green remarked in [Gre3] that for any quartic curve C in \mathbb{P}^2 there exists a projective line l that intersects C not more than in two points (an inflectional tangent to C , a bitangent, a tangent in a singular point, or a line passing through two singular points of C). Thus $\mathbb{P}^2 \setminus C$ is not hyperbolic; indeed, it contains $l \setminus C \supset \mathbb{P}^1 \setminus \{2 \text{ points}\}$, and so the Kobayashi pseudodistance $k_{\mathbb{P}^2 \setminus C}$ is degenerate along $l \setminus C$.

We do not know whether for $d \leq 2n$, $\mathcal{HE}_{n,d}$ is empty, although we know [Za3] that its complement $\mathbb{P}_{n,d} \setminus \mathcal{HE}_{n,d}$ contains a Zariski open subset.

Proposition II.1 *For a generic (in Zariski sense) hypersurface D of degree $d \leq 2n$ in \mathbb{P}^n and for any $k, 0 \leq k \leq d$, there exists a projective line l that intersects D only*

in two points with multiplicities k and $d - k$, respectively. Thus, the pseudodistance $k_{\mathbb{P}^n \setminus D}$ is degenerate along $l \setminus D$. For $d = 2n$ the number of such lines is finite.

In contrast with $\mathcal{H}_{n,d} \subseteq \mathbb{P}_{n,d}$, the subset $\mathcal{HE}_{n,d}$ is never open in the usual topology of $\mathbb{P}_{n,d}$. For instance, for any $d \geq 2n + 1$ the totally reducible hypersurfaces $C_{n,d} \in \mathcal{HE}_{n,d}$ considered above belong to the boundary of $\mathcal{HE}_{n,d}$. This follows from the next simple observation [Za4]:

Proposition II.2 *Any hypersurface D_0 in \mathbb{P}^n that contains a projective line l , is the limit of a sequence of hypersurfaces $\{D_k\}$ such that $l \cap D_k$ consists of a single point. Thus $\mathbb{P}^n \setminus D_k$ is not hyperbolic, and so $D_0 \in \overline{\mathbb{P}_{n,d} \setminus \mathcal{HE}_{n,d}}$.*

However, in [Za4] a stability principle is obtained which can be applied to fix the Kobayashi Problem II. Its proof follows the line of the proof of Theorem I.1. It gives e.g., the following result.

Theorem II.1 *Let M be a compact complex manifold and D a hypersurface in M . If D and $M \setminus D$ are both Brody hyperbolic, then $M \setminus D$ is hyperbolically embedded in M . Moreover, all the above properties are preserved by small deformations of the pair (M, D) .*

Corollary $\mathcal{HE}_{n,d} \cap \mathcal{H}_{n,d}$ is an open (but possibly empty) subset of $\mathbb{P}_{n,d}$ in the usual Hausdorff topology.

Presumably, for $d \gg n$, the intersection $\mathcal{HE}_{n,d} \cap \mathcal{H}_{n,d}$ contains a Zariski open subset of $\mathbb{P}_{n,d}$. This would imply the positive answer to the both of the Kobayashi Problems.

To construct examples of hypersurfaces in $\mathcal{HE}_{n,d} \cap \mathcal{H}_{n,d}$, one can use the following generalization of the Borel-Bloch-Cartan-Dufresnoy Theorem. It can be deduced from a result of M. Green [Gre2], and it was proven by E. Babets [Ba] by a different method.

Theorem II.2 *The complement of the union of $2n + 1$ smooth hypersurfaces in \mathbb{P}^n in general position is hyperbolically embedded into \mathbb{P}^n .*

In fact, this is true for any union of $2n + 1$ hypersurfaces such that the intersection of any $n + 1$ of them is empty (A. Eremenko and M. Sodin [ErSo]; a simplified proof has been recently done by Min Ru). Using this theorem and Theorem II.1, one can easily obtain the following

Corollary *If $\mathcal{H}_{n,k}$ is non-empty then for any $d \geq (2n + 1)k$, the set $\mathcal{HE}_{n,d} \cap \mathcal{H}_{n,d}$ is non-empty and open.*

Indeed, by Theorem II.2 the union of any $2n + 1$ smooth hyperbolic surfaces in general position belongs to $\mathcal{HE}_{n,d} \cap \mathcal{H}_{n,d}$.

In particular, from the existence of a hyperbolic surface in \mathbb{P}^3 of degree 21 [Na] it follows that $HE_{3,d} \cap H_{3,d}$ is non-empty for any $d \geq 147 = 7 \cdot 21$.

For $n = 2$ a more refined version of the Stability Principle, which uses *absorbing stratifications* [Za4], leads to the following result.

Theorem II.3 *For any $d \geq 5$ the open set $HE_{2,d} \cap H_{2,d}$ is non-empty i.e., there exists a classically open set of smooth curves in \mathbb{P}^2 of degree d with hyperbolically embedded complements.*

The bound $d \geq 5$ is sharp, as follows from the remark of M. Green mentioned above.

The first examples of smooth curves of any even degree $d \geq 30$ in $\mathcal{HE}_{2,d}$ were constructed by K. Azukawa and M. Suzuki [AzSu] by the Brody-Green method [BrGre]. Note that, if $B \subseteq \mathbb{P}^2$ is the branch curve of a regular projection to \mathbb{P}^2 of a hyperbolic

projective surface, then the complement $\mathbb{P}^2 \setminus B$ is a base of a hyperbolic covering and so is hyperbolic. But the class of such curves is rather restricted, as has been observed by F. Bogomolov, B. Moishezon and M. Teicher. For instance, the number of cusps of the branch curve of a generic projection to \mathbb{P}^2 of a smooth projective surface is divisible by 3.

In a series of papers by M. Green, J. Carlson and M. Green, H. Grauert and U. Peternell (see [Za4] for references) certain sufficient conditions were found that ensure, for an irreducible plane curve C of genus ≥ 2 , the existence of a complete Hermitian metric in the complement $\mathbb{P}^2 \setminus C$ with holomorphic sectional curvature bounded from above by a negative constant. By Ahlfors Lemma this implies that $\mathbb{P}^2 \setminus C$ is hyperbolically embedded in \mathbb{P}^2 . Any curve satisfying these conditions is singular and has degree ≥ 6 ; the only known examples are the dual curves to generic smooth plane curves of degrees $d \geq 4$.

By Green-Babets Theorem II.2, the complement to a union of 5 smooth curves in \mathbb{P}^2 in general position is hyperbolically embedded in \mathbb{P}^2 . Therefore for $d \geq 5$ the set $\mathcal{HE}_{2,d}$ contains a quasiprojective variety of positive dimension. For instance, the quasiprojective subvariety $M = \{C_{2,5}\} = \{\text{the unions of 5 lines in general position in } \mathbb{P}^2\}$ of dimension 10 is contained in $\mathcal{HE}_{2,5} \subset \mathbb{P}_{2,5} = \mathbb{P}^{20}$. Recently G. Dethloff, G. Schumacher and P.-M. Wong [DetSchWo] have shown that the complement to a union C of 4 plane curves in general position is hyperbolically embedded in \mathbb{P}^2 provided that $\deg C \geq 5$ (see P.-M. Wong's report in this volume). This fact can also be obtained by using a result of M. Green [Gre2], or the one by Y. Adachi and M. Suzuki [AdSu1] (see Theorem II.6 below).

Another result of [DetSchWo], conjectured by H. Grauert [Grau] and obtained through the value distribution theory, is the following

Theorem II.4 *a) In the space of all unions of three quadrics in \mathbb{P}^2 , there is an (explicitly defined) Zariski open subset contained in $\mathcal{HE}_{2,6}$.*

b) In the space of all unions of a line and two quadrics in \mathbb{P}^2 , there is a quasiprojective subvariety of codimension 1 contained in $\mathcal{HE}_{2,5}$.

Let us mention a related criterion of hyperbolic embedding for the complements of curves [Za2].

Proposition II.3 *Let C be a closed curve in a smooth compact complex surface M . The complement $M \setminus C$ is hyperbolically embedded in M if and only if the curve $C \setminus \text{Sing}(C)$ is hyperbolic and the complement $M \setminus C$ is Brody hyperbolic.*

The property of algebraic degeneracy of the complements of curves was treated by T. Nishino and M. Suzuki [NiSu], Y. Adachi and M. Suzuki [AdSu1,2]. The following results are worth mentioning.

Theorem II.5 ([NiSu]) *Let M and C be as above. If the logarithmic Kodaira dimension $\bar{k}(M \setminus C) = 2$, then any proper holomorphic map $f : \mathbb{C} \rightarrow M \setminus C$ is algebraically degenerate i.e., its image $f(\mathbb{C})$ is contained in a closed curve E in M .*

Theorem II.6 ([AdSu1]) *If a reducible curve C in \mathbb{P}^2 consists of at least 4 irreducible components which do not belong to the same linear pencil, then there exists a curve A in \mathbb{P}^2 such that the image of any non-constant entire curve $\mathbb{C} \rightarrow \mathbb{P}^2 \setminus C$ is contained in A . Thus, $\mathbb{P}^2 \setminus C$ has the property of strong algebraic degeneracy.*

All possible exceptions here have been completely classified. For some examples of degeneracy loci in the complements of irreducible quartic curves see [Gre3]; see also [DetShuWo] for the reducible case.

Another degeneracy principle has been used in the Babels' proof of Theorem II.2 [Ba]. It states that, *if M is a compact complex manifold and D is a normal crossings divisor in M , then, for a suitable complete Hermitian metric on $M \setminus D$, every holomorphic differential in $M \setminus D$ with logarithmic poles along D is constant on any Brody curve $\mathbb{C} \rightarrow M \setminus D$.* See also [Na] for an algebraic degeneracy principle in presence of an ample (in Siu's sense) meromorphic connection.

With evident changes, the notion of algebraic hyperbolicity can be equally applied to affine or quasiprojective algebraic varieties. This allows again to divide Problem II into two parts, likewise Problem I was divided above into Problems I.2 and I.3.

Problem II.1 *Let D be a hyperbolic hypersurface in \mathbb{P}^n such that there exists a Brody curve $\mathbb{C} \rightarrow \mathbb{P}^n \setminus D$. Is it true that there exists a rational projective curve C in \mathbb{P}^n which has not more than two places on D ?*

Problem II.2 *Let $\mathcal{L}_{n,d} \subset \mathbb{P}_{n,d}$ be the locus of all hypersurfaces D of degree d in \mathbb{P}^n which admit a rational curve C as above. Is it true that, for $d \gg n$, the complement $\mathbb{P}_{n,d} \setminus \mathcal{L}_{n,d}$ contains a Zariski open subset of $\mathbb{P}_{n,d}$? Is the locus $\mathcal{L}_{n,d}$ closed in $\mathbb{P}_{n,d}$ with its Hausdorff topology?*

Next we pass to hyperbolicity properties of the complements to hyperplanes in \mathbb{P}^n . For hyperplanes in general position, the following result is due to H. Fujimoto [Fu], M. Green [Gr], P. Kiernan and Sh. Kobayashi [KiKo].

Theorem II.7 *Let D be a union of $n+k$ hyperplanes in general position in \mathbb{P}^n , where $k > 0$. Then the image of any non-constant entire curve $\mathbb{C} \rightarrow \mathbb{P}^n \setminus D$ is contained in a linear subspace of dimension $\leq [\frac{n}{k}]$. This bound is sharp. Moreover the degeneracy locus is contained in a finite union of the 'diagonal linear subspaces' of dimension $n - k + 1$ defined by D in a canonical way. Thus $\mathbb{P}^n \setminus D$ has the property of strong algebraic degeneracy.*

For $k = 2$ this gives the upper bound $[\frac{n}{2}]$ for the dimension of the degeneracy locus. Observe that from the Borel Lemma it just follows the linear degeneracy, which means that any non-constant entire curve in the complement to $n + 2$ hyperplanes in \mathbb{P}^n in general position is contained in a hyperplane. In fact, the latter remains true without the assumption of general position [Gre1]. For $k = n + 1$, Theorem II.7 once again leads to the Borel-Bloch-Cartan-Dufresnoy Theorem.

The bound $d \geq 2n + 1$ for the hyperbolicity of $\mathbb{P}^n \setminus D$ is sharp, as is shown by the following result of V.E. Snurnitsyn [Sn], which confirms a conjecture of P. Kiernan [Ki1].

Theorem II.8 *For any union D of $2n$ hyperplanes in \mathbb{P}^n there exists a projective line which meets D at most in two points. Therefore, $\mathbb{P}^n \setminus D$ is not hyperbolic.*

Some examples of unions of hyperplanes in non-general position with hyperbolically embedded complements were given by P. Kiernan [Ki1]. In [Za2] the following conditions for a finite union D of hyperplanes in \mathbb{P}^n were considered:

(a) *There does not exist a pair of points x, y in \mathbb{P}^n such that each hyperplane in D passes through at least one of these points. In other words, there does not exist a*

projective line $l = (x, y)$ which intersects the union of all those hyperplanes in D that do not contain l , in at most two points.

(b) There does not exist a pair of points (x, y) in \mathbb{P}^n such that each hyperplane in D passes through exactly one of these points. In other words, there does not exist a projective line $l = (x, y)$ that intersects D in at most two points.

If condition (b) fails then, clearly, the Kobayashi pseudodistance $k_{\mathbb{P}^n \setminus D}$ is degenerate along l . If (a) is violated then the limit of $k_{\mathbb{P}^n \setminus D}$ is degenerate along l . The following criteria were obtained in [Za2, Sect.3].

Theorem II.9 *Let D be as above. The complement $\mathbb{P}^n \setminus D$ is hyperbolically embedded in \mathbb{P}^n if and only if (a) holds. It is Picard hyperbolic if and only if (b) is fulfilled. Furthermore, for $n = 2$ (b) is equivalent to the hyperbolicity of $\mathbb{P}^2 \setminus D$.*

The latter had been conjectured by S. Iitaka.

Another criterion for the Picard hyperbolicity of complements of hyperplanes has been recently obtained by Min Ru [Ru].

Theorem II.10 *The complement $\mathbb{P}^n \setminus D$ of a finite union D of hyperplanes in \mathbb{P}^n is Picard hyperbolic if and only if, for any linear subspace V in \mathbb{P}^n which is not contained in D , the intersection $V \cap D$ contains at least three distinct hyperplanes of V that are linearly dependent.*

The latter condition is obviously equivalent to (b). An algorithm that allows to check it is given in [Ru]. To verify (b) one can equally apply an algorithm similar to the simplex method, which consists in passing from one pair of isolated intersection points of n hyperplanes in D (if there is any such pair) to another one.

In conclusion, let us mention the Lang Conjecture on equivalence of Picard hyperbolicity and mordelleness (see [La]). For the complements of hyperplanes, it was proven by P.-M. Wong and M. Ru [WoRu] under the assumption of general position, and by M. Ru [Ru] without this assumption.

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